On the scope of validity of the norm limitation theorem for quasilocal fields

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1. Introduction

This paper is devoted to the study of norm groups of the fields pointed out in the title, i.e. of fields whose finite extensions are primarily quasilocal (briefly, PQL). It concentrates on the special case where the considered ground fields are strictly quasilocal, i.e. their finite extensions are strictly PQL (or equivalently, these extensions admit one-dimensional local class field theory, see [7]). The paper shows (see Theorem 1.1) that if E is a quasilocal field, R/E is a finite separable extension, and R_{ab} is the maximal abelian subextension of E in R, then the norm groups $\mathrm{N}(\mathrm{R}/\mathrm{E})$ and $\mathrm{N}(\mathrm{R}_{\mathrm{ab}}/\mathrm{E})$ are equal, provided that the natural Brauer group homomorphism $\mathrm{Br}(\mathrm{E}) \to \mathrm{Br}(\mathrm{L})$ is surjective, for every finite extension L of E. This is established in a more general form used in [10] (see also (5.2) (i)) for describing the norm groups of finite separable extensions of strictly quasilocal fields with Henselian discrete valuations. Relying on [10], we prove here that Theorem 1.1 and the main results of [9], stated as (1.1) (ii), determine to a considerable extent the scope of validity of the classical norm limitation theorem (cf. [11, Ch. 6, Theorem 8]), in the case of strictly PQL ground fields. The present research also sheds light on the possibility of reducing the study of norm groups of quasilocal fields to the special case of finite abelian extensions.

The basic field-theoretic notions needed for describing the main results of this paper are the same as those in [9]. As usual, E^* denotes the multiplicative group of a field E. We say that E is formally real, if -1 is not presentable as a finite sum of squares of elements of E; the field E is called nonreal, otherwise. For convenience of the reader, we recall that E is said to be a PQL-field, if every cyclic extension F of E is embeddable as an E-subalgebra in each central division E-algebra D of Schur index $\mathrm{ind}(\mathrm{D})$ divisible by the degree $[\mathrm{F}\colon\mathrm{E}]$. When this occurs, we say that E is

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strictly PQL, if the p-component $\mathrm{Br}(\mathrm{E})_p$ of the Brauer group $\mathrm{Br}(\mathrm{E})$ is nontrivial in case p runs through the set $\mathrm{P}(\mathrm{E})$ of those prime numbers, for which E is properly included in its maximal p-extension $\mathrm{E}(p)$ in a separable closure $\mathrm{E}_{\mathrm{sep}}$ of E. It is worth noting that PQL-fields and quasilocal fields appear naturally in the process of characterizing some of the basic types of stable fields with Henselian valuations (see [8] and the references there). Our research, however, is primarily motivated by the fact that strictly PQL-fields admit local class field theory, and by the validity of the converse in all presently known cases (see [7, Theorem 1 and Sect. 2]). As to the choice of our main topic, it is determined to a considerable extent by the following results:

(1.1) (i) $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\mathrm{R}_{\mathrm{ab}}/\mathrm{E})$, provided that R is a finite separable extension of a field E possessing a Henselian discrete valuation with a quasifinite residue field $\widehat{\mathrm{E}}$ [18] (see also [24] and [29]]);
(ii) $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\mathrm{R}_{\mathrm{ab}}/\mathrm{E})$ in case E is a PQL-field and R is an intermediate field of a finite Galois extension M/E with a nilpotent Galois group; for each nonnilpotent finite group G, there exists an algebraic extension $\mathrm{E}(\mathrm{G})$ of $\mathbb{Q}$, which is strictly PQL and has a Galois extension $\mathrm{M}(\mathrm{G})$, such that $\mathrm{G}(\mathrm{M}(\mathrm{G})/\mathrm{E}(\mathrm{G}))$ is isomorphic to G and $\mathrm{N}(\mathrm{M}(\mathrm{G})/\mathrm{E}(\mathrm{G}))$ is a proper subgroup of $\mathrm{N}(\mathrm{M}(\mathrm{G})_{\mathrm{ab}}/\mathrm{E}(\mathrm{G}))$ [9, Theorems 1.1 and 1.2];
(iii) If E is an algebraic strictly PQL-extension of a global field E_0, and R/E is a finite extension, then $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\Phi(\mathrm{R})/\mathrm{E})$, for some abelian finite extension $\Phi(\mathrm{R})$ of E, which is uniquely determined by R/E, up-to an E-isomorphism (see the references after the statement of [9, Theorem 1.2]).

The main purpose of this paper is to shed an additional light on these facts by proving the following two statements (the former of which generalizes (1.1) (i), see also Remark 4.4, for more details):

Theorem 1.1. *Let* E *be a quasilocal field and* R/E *a finite separable extension. Then* $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\mathrm{R}_{\mathrm{ab}}/\mathrm{E})$ *in the following two special cases:*
(i) *The natural homomorphism of* $\mathrm{Br}(\mathrm{E})$ *into* $\mathrm{Br}(\mathrm{L})$ *is surjective, for every finite extension* L *of* E*;*
(ii) *There exists an abelian finite extension* $\Phi(\mathrm{R})$ *of* E*, such that* $\mathrm{N}(\Phi(\mathrm{R})/\mathrm{E}) = \mathrm{N}(\mathrm{R}/\mathrm{E})$*.*

Theorem 1.2. *There exists a strictly quasilocal nonreal field* E *satisfying the following conditions:*
(i) *the absolute Galois group* $\mathrm{G}_{\mathrm{K}} := \mathrm{G}(\mathrm{K}_{\mathrm{sep}}/\mathrm{K})$ *is not pronilpotent;*
(ii) *every finite extension* R *of* K *is subject to the following alternative:*

(α) R *is an intermediate field of a finite Galois extension* M(R)/K *with a nilpotent Galois group;*
(β) N(R/K) *does not equal the norm group of any abelian finite extension of* K *.*

In addition to (1.1) and Theorems 1.1 and 1.2, it has been proved in [6] that the description of the norm groups of finite separable extensions of a strictly PQL-field F does not reduce to the study of Galois extensions M/F with G(M/F) belonging to any given proper class of finite groups, which is closed under the formation of subgroups, quotient groups and group extensions. Also, it has been shown in [5] that a formally real strictly quasilocal field E has the properties required by Theorem 1.2 (i) and (ii) unless it is real closed.

Throughout the paper, simple algebras are supposed to be associative with a unit and finite-dimensional over their centres, and Galois groups are viewed as profinite with respect to the Krull topology. For each simple algebra A, we consider only subalgebras of A containing its unit. Our basic terminology and notation concerning valuation theory, simple algebras and Brauer groups are standard (for example, as in [12; 15; 36] and [20], as well as those related to profinite groups, Galois cohomology, field extensions and Galois theory (see, for example, [25; 13] and [15]). We refer the reader to [28, Sect. 1] and [4, Sect. 2], for the definitions of a symbol algebra and of a symbol p-algebra (see also [26, Ch. XIV, Sects. 2 and 5]).

Here is an overview of the paper: Section 2 includes preliminaries used in the sequel. Theorems 1.1 and 1.2 are proved in Sections 3-4 and 5, respectively. Section 5 contains a characterization of the fields singled out by Theorem 1.2 among those endowed with a Henselian discrete valuation and possessing the strictly PQL-property.

2. Preliminaries

Let E be a field, Nr (E) the set of norm groups of finite extensions of E in $\mathrm{E}_{\mathrm{sep}}$, and $\Omega(\mathrm{E})$ the set of finite abelian extensions of E in $\mathrm{E}_{\mathrm{sep}}$. We say that E admits (one-dimensional) local class field theory, if the mapping π of $\Omega(\mathrm{E})$ into Nr (E) defined by the rule $\pi(\mathrm{F}) = \mathrm{N}(\mathrm{F}/\mathrm{E})$: $\mathrm{F} \in \Omega(\mathrm{E})$, is injective and satisfies the following two conditions, for each pair $(\mathrm{M}_1, \mathrm{M}_2) \in \Omega(\mathrm{E}) \times \Omega(\mathrm{E})$:
The norm group of the compositum $\mathrm{M}_1\mathrm{M}_2$ is equal to the intersection $\mathrm{N}(\mathrm{M}_1/\mathrm{E}) \cap \mathrm{N}(\mathrm{M}_2/\mathrm{E})$ and $\mathrm{N}((\mathrm{M}_1 \cap \mathrm{M}_2)/\mathrm{E})$ equals the inner group product $\mathrm{N}(\mathrm{M}_1/\mathrm{E})\mathrm{N}(\mathrm{M}_2/\mathrm{E})$.
We call E a field with (one-dimensional) local p-class field theory, for some prime p, if the restriction of π on the set of finite abelian extensions of E in $\mathrm{E}(p)$ has the same properties. Our approach to the study of fields with such a theory is based on the following two lemmas (proved in [9]).

Lemma 2.1. *Let* E *be a field and* L *an extension of* E *presentable as a compositum of extensions* L_1 *and* L_2 *of* E *of relatively prime degrees. Then* $\mathrm{N}(\mathrm{L}/\mathrm{E}) = \mathrm{N}(\mathrm{L}_1/\mathrm{E}) \cap \mathrm{N}(\mathrm{L}_2/\mathrm{E})$, $\mathrm{N}(\mathrm{L}_1/\mathrm{E}) = \mathrm{E}^* \cap \mathrm{N}(\mathrm{L}/\mathrm{L}_2)$, *and there is a group isomorphism* $\mathrm{E}^*/\mathrm{N}(\mathrm{L}/\mathrm{E}) \cong (\mathrm{E}^*/\mathrm{N}(\mathrm{L}_1/\mathrm{E})) \times (\mathrm{E}^*/\mathrm{N}(\mathrm{L}_2/\mathrm{E}))$.

Lemma 2.2. *Let* E *be a field,* M *a finite Galois extension of* E *with a nilpotent Galois group* $\mathrm{G}(\mathrm{M}/\mathrm{E})$, R *an intermediate field of* M/E *not equal to* E , Π *the set of prime numbers dividing* $[\mathrm{R}\colon\mathrm{E}]$, M_p *the maximal* p*-extension of* E *in* M , *and* $\mathrm{R}_p = \mathrm{R} \cap \mathrm{M}_p$, *for each* $p \in \Pi$. *Then:*
(i) R *is equal to the compositum of the fields* $\mathrm{R}_p\colon\ p \in \Pi$, *and* $[\mathrm{R}\colon\mathrm{E}] = \prod_{p\in\Pi}[\mathrm{R}_p\colon\mathrm{E}]$ *;*
(ii) $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \cap_{p\in\Pi}\mathrm{N}(\mathrm{R}_p/\mathrm{E})$ *and the quotient group* $\mathrm{E}^*/\mathrm{N}(\mathrm{R}/\mathrm{E})$ *is isomorphic to the direct product of the groups* $\mathrm{E}^*/\mathrm{N}(\mathrm{R}_p/\mathrm{E})\colon\ p \in \Pi$.

It is clear from Lemma 2.2 that a field E admits local class field theory if and only if it admits local p-class field theory, for every $p \in \mathrm{P}(\mathrm{E})$. Our next lemma, proved in [8, Sect. 4], shows that $\mathrm{Br}\,(\mathrm{E})_p \neq \{0\}$ whenever E is a field with such a theory, for a given $p \in \mathrm{P}(\mathrm{E})$.

Lemma 2.3. *Let* E *be a field, such that* $\mathrm{Br}\,(\mathrm{E})_p = \{0\}$, *for some prime number* p. *Then* $\mathrm{Br}\,(\mathrm{E}_1)_p = \{0\}$ *and* $\mathrm{N}(\mathrm{E}_1/\mathrm{E}) = \mathrm{E}^*$, *for every finite extension* E_1 *of* E *in* $\mathrm{E}(p)$.

The following lemma is known (cf. [25, Ch. II, 2.3 and 3.1]) and plays an essential role in the proof of Theorem 1.1.

Lemma 2.4. *For a field* E *and a prime number* p , *the following conditions are equivalent:*
(i) $\mathrm{Br}\,(\mathrm{E}')_p = \{0\}$, *for every algebraic extension* E' *of* E *;*
(ii) *The exponent of the group* $\mathrm{E}_1^*/\mathrm{N}(\mathrm{E}_2/\mathrm{E}_1)$ *is not divisible by* p , *for any pair* $(\mathrm{E}_1, \mathrm{E}_2)$ *of finite extensions of* E *in* $\mathrm{E}_{\mathrm{sep}}$, *such that* $\mathrm{E}_1 \subseteq \mathrm{E}_2$.

For a detailed proof of Lemma 2.4, we refer the reader to [5]. Let now Φ be a field and Φ_p the extension of Φ in Φ_{sep} generated by a primitive p-th root of unity ε_p , for some prime p . It is well-known (cf. [15, Ch. VIII, Sect. 3]) that then Φ_p/Φ is a cyclic extension of degree $[\Phi_p\colon\Phi] := m$ dividing $p-1$. Denote by φ some Φ-automorphism of Φ_p of order m , fix an integer s so that $\varphi(\varepsilon_p) = \varepsilon_p^s$, and put $\mathrm{V}_i = \{\alpha_i \in \Phi_p^*\colon\ \varphi(\alpha_i)\alpha_i^{-s^i} \in \Phi_p^{*p}\}$ and $\overline{\mathrm{V}}_i = \mathrm{V}_i/\Phi_p^{*p}\colon\ i = 0, \dots, m-1$. Clearly, the quotient group $\Phi_p^*/\Phi_p^{*p} := \overline{\Phi}_p$ can be viewed as a vector space over the field $\mathbb{F}_p$ with p elements. Considering the linear operator $\bar{\varphi}$ of $\overline{\Phi}_p$, defined by the rule $\bar{\varphi}(\alpha\Phi_p^{*p}) = \varphi(\alpha)\Phi_p^{*p}\colon\ \alpha \in \Phi_p^*$, and taking into account that the subspace of $\overline{\Phi}_p$, spanned by its elements $\bar{\varphi}^i(\bar{\alpha})\colon\ i = 0, \dots, m-1$, is finite-dimensional and

$\bar{\varphi}$-invariant, for each $\bar{\alpha} \in \overline{\Phi}_p$, one obtains from Maschke's theorem the following statement:

(2.1) The sum of the subspaces $\overline{\mathrm{V}}_i$: $i = 0, \dots, m-1$ is direct and equal to $\overline{\Phi}_p$.

Let L be an extension of Φ_p in Φ_{sep}, obtained by adjoining a p-th root η_p of an element $\beta \in (\Phi_p^* \setminus \Phi_p^{*p})$. It is clear from Kummer's theory that $[\mathrm{L}\colon \Phi] = pm$ and the following assertions hold true:

(2.2) L/Φ is a Galois extension if and only if $\beta \in \mathrm{V}_j$, for some index j. Such being the case, every Φ_p-automorphism ψ of L of order p satisfies the equality $\varphi'\psi\varphi'^{-1} = \psi^{s'}$, where $s' = s^{1-j}$ and φ' is an arbitrary automorphism of L extending φ. Moreover, L and Φ are related as follows:
(i) L/Φ is cyclic if and only if $\beta \in \mathrm{V}_1$ (Albert [1, Ch. IX, Theorem 6]);
(ii) L is a root field over Φ of the binomial $\mathrm{X}^p - a$, for some $a \in \Phi^*$, if and only if $\beta \in \mathrm{V}_0$, i.e. $s' = s$; when this occurs, one can take as a the norm $\mathrm{N}_{\Phi}^{\Phi_p}(\beta)$.

Statements (2.1), (2.2) and the following observations will be used for proving Theorems 1.1 and 1.2.

(2.3) For a symbol Φ_p-algebra $\mathrm{A}_{\varepsilon_p}(\alpha, \beta; \Phi_p)$ (of dimension p^2), where $\alpha \in \Phi_p^*$ and $\beta \in \mathrm{V}_j \setminus \Phi_p^{*p}$, the following conditions are equivalent:
(i) $\mathrm{A}_{\varepsilon_p}(\alpha, \beta; \Phi_p)$ is Φ_p-isomorphic to $\mathrm{D} \otimes_{\Phi} \Phi_p$, for some central simple Φ-algebra D;
(ii) If $\alpha = \prod_{i=0}^{m-1} \alpha_i$ and $\alpha_i \in \mathrm{V}_i$, for each index i, then $\mathrm{A}_{\varepsilon_p}(\alpha, \beta; \Phi_p)$ is isomorphic to the symbol Φ_p-algebra $\mathrm{A}_{\varepsilon_p}(\alpha_{j'}, \beta; \Phi_p)$, where j' is determined so that m divides $j' + j - 1$;
(iii) With notations being as in (ii), $\alpha_i \in \mathrm{N}(\mathrm{L}/\Phi_p)$: $i \neq j'$.

The main results of [7, Sect. 2] and [8] used in the present paper (sometimes without an explicit reference) can be stated as follows:

Proposition 2.5. *Let* E *be a strictly* p*-quasilocal field, for some* $p \in \mathrm{P}(\mathrm{E})$. *Assume also that* R *is a finite extension of* E *in* $\mathrm{E}(p)$, *and* D *is a central division* E*-algebra of* p*-primary dimension. Then* R, E *and* D *have the following properties:*
(i) R *is a* p*-quasilocal field and* $\mathrm{ind}(\mathrm{D}) = \exp(\mathrm{D})$;
(ii) $\mathrm{Br}(\mathrm{R})_p$ *is a divisible group unless* $p = 2$, $\mathrm{R} = \mathrm{E}$ *and* E *is formally real; in the noted exceptional case,* $\mathrm{Br}(\mathrm{E})_2$ *is of order* 2 *and* $\mathrm{E}(2) = \mathrm{E}(\sqrt{-1})$;
(iii) E *admits local* p*-class field theory, provided that* $Br(\mathrm{E})_p \neq \{0\}$;
(iv) R *embeds in* D *as an* E*-subalgebra if and only if* $[\mathrm{R}\colon \mathrm{E}]$ *divides* $\mathrm{ind}(\mathrm{D})$.

3. p-primary analogue to Theorem 1.1

Let E be a field, R/E a finite separable extension, and for each prime p, let $\mathrm{R}_{\mathrm{ab},p}$ be the maximal abelian p-extension of E in R, ρ_p the greatest integer dividing $[\mathrm{R}\colon\mathrm{E}]$ and not divisible by p, and $\mathrm{N}_p(\mathrm{R}/\mathrm{E})$ the set of those elements $u_p \in \mathrm{E}^*$, for which the co-set $u_p\mathrm{N}(\mathrm{R}/\mathrm{E})$ is a p-element of the group $\mathrm{E}^*/\mathrm{N}(\mathrm{R}/\mathrm{E})$. Clearly, $u^{\rho_p} \in \mathrm{N}_p(\mathrm{R}/\mathrm{E})$, for every $u \in \mathrm{E}^*$. Observing also that $u^{\rho_p} \in \mathrm{N}(\mathrm{R}_{\mathrm{ab},p}/\mathrm{E})$ whenever $u \in \mathrm{N}(\mathrm{R}_{\mathrm{ab}}/\mathrm{E})$ and p is prime, one concludes that Theorem 1.1 (i) can be deduced from its p-primary analogue stated as follows:

Theorem 3.1. *Assume that* E *is a quasilocal field, such that the natural homomorphism of* $\mathrm{Br}(\mathrm{E})$ *into* $\mathrm{Br}(\mathrm{L})$ *maps* $\mathrm{Br}(\mathrm{E})_p$ *surjectively on* $\mathrm{Br}(\mathrm{L})_p$, *for some prime number* p *and every finite extension* L *of* E. *Then* $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\mathrm{R}_{\mathrm{ab},p}/\mathrm{E}) \cap \mathrm{N}_p(\mathrm{R}/\mathrm{E})$, *for each finite extension* R *of* E *in* $\mathrm{E}_{\mathrm{sep}}$.

In what follows, up-to the end of the next Section, our main objective is to prove Theorems 3.1 and 1.1. Evidently, $\mathrm{N}(\mathrm{R}/\mathrm{E}) \subseteq (\mathrm{N}_p(\mathrm{R}/\mathrm{E}) \cap \mathrm{N}(\mathrm{R}_{\mathrm{ab}}/\mathrm{E}))$, so we have to prove that $\mathrm{N}_p(\mathrm{R}/\mathrm{E}) \cap \mathrm{N}(\mathrm{R}_{\mathrm{ab}}/\mathrm{E})$ is a subgroup of $\mathrm{N}(\mathrm{R}/\mathrm{E})$. Our assumptions show that if $\mathrm{Br}(\mathrm{E})_p = \{0\}$, then $\mathrm{Br}(\mathrm{L})_p = \{0\}$, for every finite extension L of E, which reduces our assertion to a consequence of Lemma 2.4. Assuming further that $\mathrm{Br}(\mathrm{E})_p \neq \{0\}$ and $\mathbb{F}_p$ is a field with p elements (identifying it with the prime subfield of E, in the case of $\mathrm{char}(\mathrm{E}) = p$), we prove in the rest of this Section the validity of Theorem 3.1 in the special case where R/E is a normal extensions with a solvable Galois group. The main part of our argument is presented by the following two lemmas.

Lemma 3.2. *Let* E *be a field and* p *a prime number satisfying the conditions of Theorem 3.1, and let* M/E *be a Galois extension with* $\mathrm{G}(\mathrm{M}/\mathrm{E})$ *satisfying the following conditions:*
(i) $\mathrm{G}(\mathrm{M}/\mathrm{E})$ *is nonabelian and isomorphic to a semidirect product* $\mathrm{E}_{p;k} \times \mathrm{C}_\pi$ *of an elementary abelian* p*-group of order* p^k *by a group* C_π *of prime order* π *not equal to* p, *where* k *is the minimal positive integer solution to the congruence* $p^k \equiv 1$ $(\mathrm{mod}\pi)$;
(ii) $\mathrm{E}_{p;k}$ *is a minimal normal subgroup of* $\mathrm{G}(\mathrm{M}/\mathrm{E})$.
Then $\mathrm{N}(\mathrm{M}/\mathrm{E}_1)$ *includes* E^*, *where* E_1 *is the intermediate field of* M/E *corresponding by Galois theory to* $\mathrm{E}_{p;k}$.

Proof. Our assumptions indicate that E_1/E is a cyclic extension of degree π, and under the additional hypothesis that $\mathrm{Br}(\mathrm{E})_p \neq \{0\}$, this means that $\mathrm{Br}(\mathrm{E}_1)_p \neq \{0\}$ (see [20, Sect. 13.4]). Therefore, by Proposition 2.5 (iii), E_1 admits local p-class field theory, so it is sufficient to show that $\mathrm{E}^* \subseteq \mathrm{N}(\mathrm{M}_1/\mathrm{E}_1)$, for every cyclic

extension M_1 of E_1 in M. Suppose first that E contains a primitive p-th root of unity or $\mathrm{char}\,(\mathrm{E}) = p$, and fix an E-automorphism ψ of E_1 of order π. As $\mathrm{G}(\mathrm{M}/\mathrm{E}_1)$ is an elementary abelian p-group of rank k, Kummer's theory and the Artin-Schreier theorem (cf. [15, Ch. VIII, Sect. 6]) imply the existence of a subset $\mathrm{S} = \{\rho_j\colon\ j = 1, \dots, k\}$ of E_1, such that the root field over E_1 of the polynomial set $\{f_j(\mathrm{X}) = \mathrm{X}^p - u\mathrm{X} - \rho_j\colon\ j = 1, \dots, k\}$ equals M, where $u = 1$, if $\mathrm{char}\,(\mathrm{E}) = p$, and $u = 0$, otherwise. For each index j, denote by z_j the element $\psi(u_j)u_j^{-1}$ in case E contains a primitive p-th root of unity, and put $z_j = \psi(u_j) - u_j$, if $\mathrm{char}\,(\mathrm{E}) = p$. Note that M is a root field over E_1 of the set of polynomials $\{g_j(\mathrm{X}) = \mathrm{X}^p - u\mathrm{X} - z_j\colon\ j = 1, \dots, k\}$. This can be deduced from the following two statements:

(3.1) (i) If $\mathrm{char}\,(\mathrm{E}) = p$, $r(\mathrm{E}_1) = \{\lambda^p - \lambda\colon\ \lambda \in \mathrm{E}_1\}$ and $\mathrm{M}(\mathrm{E}_1)$ is the additive subgroup of E generated by the union $\mathrm{S} \cup r(\mathrm{E}_1)$, then $r(\mathrm{E}_1)$ and $\mathrm{M}(\mathrm{E}_1)$ are ψ-invariant, regarded as vector spaces over $\mathbb{F}_p$; moreover, the linear operator of the quotient space $\mathrm{M}(\mathrm{E}_1)/r(\mathrm{E}_1)$, induced by $\psi - id_{\mathrm{E}_1}$ is an isomorphism;
(ii) If E contains a primitive p-th root of unity, $\mathrm{M}(\mathrm{E}_1)$ is the multiplicative subgroup of E_1^* generated by the union $\mathrm{S} \cup \mathrm{E}_1^{*p}$, and the mapping $\psi_1\colon\ \mathrm{E}_1^*/\mathrm{E}_1^{*p} \to \mathrm{E}_1^*/\mathrm{E}_1^{*p}$ is defined by the rule $\psi_1(\alpha\mathrm{E}_1^{*p}) = \psi(\alpha)\alpha^{-1}\mathrm{E}_1^{*p}\colon\ \alpha \in \mathrm{E}_1^*$, then ψ_1 is a linear operator of $\mathrm{E}_1^*/\mathrm{E}_1^{*p}$ (regarded as a vector space over $\mathbb{F}_p$), $\mathrm{M}(\mathrm{E}_1)/\mathrm{E}_1^{*p}$ is a k-dimensional ψ_1-invariant subspace of $\mathrm{E}_1^*/\mathrm{E}_1^{*p}$, and the linear operator of $\mathrm{M}(\mathrm{E}_1)/\mathrm{E}_1^{*p}$ induced by ψ_1 is an isomorphism.

Most of the assertions of (3.1) are well-known. One should, possibly, only note here that the concluding parts of (3.1) (i) and (3.1) (ii) follow from the fact that $\mathrm{G}(\mathrm{M}/\mathrm{E}_1)$ is the unique normal proper subgroup of $\mathrm{G}(\mathrm{M}/\mathrm{E})$, and by Galois theory, this means that E_1 is the unique normal proper extension of E in M. The obtained result implies the nonexistence of a cyclic extension of E in M of degree p, which enables one to deduce from Kummer's theory and the Artin-Schreier theorem the triviality of the kernels of the considered linear operators. Thus our argument leads to the conclusion that the discussed special case of Lemma 3.2 will be proved, if we establish the validity of the following two statements, for each index j:

(3.2) (i) If E contains a primitive p-th root of unity ε and c is an element of E^*, then the symbol E_1-algebra $\mathrm{A}_\varepsilon(z_j, c; \mathrm{E}_1)$ is trivial;
(ii) If $\mathrm{char}\,(\mathrm{E}) = p$ and $c \in \mathrm{E}^*$, then the p-symbol E-algebra $\mathrm{E}[z_j, c)$ is trivial.

Denote by D_j the symbol p-algebra $\mathrm{E}_1[\rho_j, c)$, if $\mathrm{char}\,(\mathrm{E}) = p$, and the symbol E_1-algebra $\mathrm{A}_\varepsilon(\rho_j, c; \mathrm{E}_1)$ in case E_1 contains a primitive p-th root of unity ε. It follows from the assumptions of Theorem 3.1 that D_j is isomorphic over E_1 to $\Delta_j \otimes_{\mathrm{E}} \mathrm{E}_1$, for some central division E-algebra Δ_j. This implies that ψ is extendable to an

automorphism $\bar{\psi}$ of D_j, regarded as an algebra over E. Thus it becomes clear that D_j is E_1-isomorphic to $E_1[\psi(\rho_j), c)$ or $A_\varepsilon(\psi(\rho_j), c; E_1)$ depending on whether or not char $(E) = p$. Applying now the general properties of local symbols (cf. [26, Ch. XIV, Propositions 4 and 11]), one proves (3.2).

It remains for us to prove Lemma 3.2, assuming that $p \neq$ char (E) and E does not contain a primitive p-th root of unity. Let ε be such a root in M_{sep}. It is easily verified that if $E(\varepsilon) \cap E_1 = E$, then $M(\varepsilon)/E(\varepsilon)$ is a Galois extension, such that $G((M(\varepsilon)/E(\varepsilon))$ is canonically isomorphic to $G(M/E)$. Since $E(\varepsilon)$ and p satisfy the conditions of the lemma, our considerations prove in this case that $E(\varepsilon)^* \subseteq N(M(\varepsilon)/E_1(\varepsilon))$. Hence, by Lemma 2.1, applied to the triple $(E_1, M, E_1(\varepsilon))$ instead of (E, L_1, L_2), we have $E^* \subseteq N(M/E_1)$, which reduces the proof of Lemma 3.2 to the special case in which E_1 is an intermediate field of $E(\varepsilon)/E$. Fix a generator φ of $G(E(\varepsilon)/E)$, and an integer s so that $\varphi(\varepsilon) = \varepsilon^s$. Observing that M/E is a noncyclic Galois extension of degree $p\pi$, one obtains from (2.2) and the cyclicity of M over E_1 that $M(\varepsilon)$ is generated over $E(\varepsilon)$ by a p-th root of an element ρ of $E(\varepsilon)$ with the property that $\varphi(\rho)\rho^{-s'} \in E(\varepsilon)^{*p}$, where s' is a positive integer such that $s'^{\pi} \equiv s^{\pi} \pmod{p}$ and $s' \not\equiv s \pmod{p}$. It is therefore clear from (2.3), the surjectivity of the natural homomorphism of $\mathrm{Br}(E)_p$ into $\mathrm{Br}(E(\varepsilon))_p$, and [20, Sect. 15.1, Proposition b] that $A_\varepsilon(\rho, c; E(\varepsilon))$ is isomorphic to the matrix $E(\varepsilon)$-algebra $M_p(E(\varepsilon))$, for every $c \in E^*$. One also sees that $E^* \subseteq N(M(\varepsilon)/E(\varepsilon))$. As $[M\colon E_1] = p$ and $[E(\varepsilon)\colon E_1]$ divides $(p-1)/\pi$, Lemma 2.1 ensures now that $E^* \subseteq N(M/E_1)$, so Lemma 3.2 is proved.

Lemma 3.3. *Assume that* E *is a quasilocal field whose finite extensions satisfy the conditions of Theorem 3.1, for a given prime number* p, *and suppose that* M/E *is a finite Galois extension, such that* $G(M/E)$ *is a solvable group. Then* $N_p(M/E) \cap N(M_{\mathrm{ab},p}/E)$ *is a subgroup of* $N(M/E)$.

Proof. It is clearly sufficient to prove the lemma under the hypothesis that $N(M'/E')$ includes $N_p(M'/E') \cap N(M'_{\mathrm{ab},p}/E')$, provided that E' and p satisfy the conditions of Theorem 3.1, and M'/E' is a Galois extension with a solvable Galois group of order less than $[M\colon E]$. As in the proof of [9, Theorem 1.1], we first show that then one may assume further that $G(M/E)$ is a Miller-Moreno group (i.e. nonabelian with abelian proper subgroups). Our argument relies on the fact that the class of fields satisfying the conditions of Theorem 3.1 is closed under the formation of finite extensions. Note that if $G(M/E)$ is not Miller-Moreno, then it possesses a nonabelian subgroup H whose commutator subgroup $[H, H]$ is normal in $G(M/E)$. Indeed, one can take as H the commutator subgroup $[G(M/E), G(M/E)]$ in case $G(M/E)$ is not metabelian, and suppose that H is

any nonabelian maximal subgroup of $\mathrm{G(M/E)}$, otherwise. Denote by F and L the intermediate fields of $\mathrm{M/E}$ corresponding to H and $[\mathrm{H},\mathrm{H}]$, respectively. Our choice of H and Galois theory indicate that $\mathrm{L/E}$ is a Galois extension such that $\mathrm{M_{ab}} \subseteq \mathrm{L}$ and $\mathrm{E} \neq \mathrm{L} \neq \mathrm{M}$, so our additional hypothesis and Lemma 2.2 lead to the conclusion that $\mathrm{N}_p(\mathrm{L/E}) \cap \mathrm{N}(\mathrm{M}_{\mathrm{ab},p}/\mathrm{E}) = \mathrm{N}_p(\mathrm{L/E}) \cap \mathrm{N}(\mathrm{M_{ab}/E}) \subseteq \mathrm{N(L/E)}$ and $\mathrm{N}_p(\mathrm{M/F}) \cap \mathrm{N(L/F)} \subseteq \mathrm{N(M/F)}$. Let now μ be an element of $\mathrm{N}_p(\mathrm{M/E}) \cap \mathrm{N}(\mathrm{M_{ab,p}/E})$, and $\lambda \in \mathrm{L}^*$ a solution to the norm equation $\mathrm{N}_{\mathrm{E}}^{\mathrm{L}}(\mathrm{X}) = \mu$. Then one can find an integer k not divisible by p and such that $\mathrm{N}_{\mathrm{F}}^{\mathrm{L}}(\lambda)^k \in \mathrm{N}_p(\mathrm{M/F})$. It is therefore clear that $\mathrm{N}_{\mathrm{F}}^{\mathrm{L}}(\lambda)^k \in \mathrm{N(M/F)}$ and $\mu^k \in \mathrm{N(M/E)}$. As $\mu \in \mathrm{N}_p(\mathrm{M/E})$, this implies that $\mu \in \mathrm{N(M/E)}$, which yields the desired reduction. In view of the former part of (1.1) (ii), one may also assume that $\mathrm{G(M/E)}$ is a nonnilpotent Miller-Moreno group. The assertion of Lemma 3.3 is obvious, if p does not divide the order $o([\mathrm{G},\mathrm{G}])$ of $[\mathrm{G},\mathrm{G}]$, so we suppose further that $p \mid o([\mathrm{G},\mathrm{G}])$. By the classification of these groups [17] (cf. also [22, Theorem 445]), this means that $\mathrm{G(M/E)}$ has the following structure:

(3.3) (i) $\mathrm{G(M/E)}$ is isomorphic to a semi-direct product $\mathrm{E}_{p;k} \times \mathrm{C}_{\pi^n}$ of $\mathrm{E}_{p;k}$ by a cyclic group C_{π^n} of order π^n, for some different prime numbers p and π, where k satisfies condition (i) of Lemma 3.2;
(ii) $\mathrm{E}_{p;k}$ is a minimal normal subgroup of $\mathrm{G(M/E)}$, $\mathrm{E}_{p;k} = [\mathrm{G(M/E)},\ \mathrm{G(M/E)}]$ and the centre of $\mathrm{G(M/E)}$ equals the subgroup $\mathrm{C}_{\pi^{n-1}}$ of C_{π^n} of order π^{n-1}.

It follows from (3.2) and Galois theory that $\mathrm{M}_{ab}/\mathrm{E}$ is cyclic of degree π^n. This yields $\mathrm{N}_{\mathrm{M}_{ab}}^{\mathrm{M}}(\eta) = \eta^{p^k}$, for every $\eta \in \mathrm{M}_{ab}$, and thereby, implies that $c^{p^k} \in \mathrm{N(M/E)}$ in case $c \in \mathrm{N(M_{ab}/E)}$. It is therefore clear from the equality g.c.d. $(p^k, \pi^n) = 1$ that Lemma 3.3 will be proved, if we show that $c^{\pi^n} \in \mathrm{N(M/E)}$ whenever $c \in \mathrm{E}^*$. By Lemma 3.2, if $n = 1$, then M^* contains an element ξ of norm c over $\mathrm{E}_1 = \mathrm{M}_{ab}$, which means that $\mathrm{N}_{\mathrm{E}}^{\mathrm{M}}(\xi) = c^\pi$. Suppose now that $n \geq 2$, put $\tilde{\pi} = \pi^{n-1}$, denote by $\mathrm{C}_{\tilde{\pi}}$ the subgroup of $\mathrm{G(M/E)}$ of order $\tilde{\pi}$, and let M' and E' be the intermediate fields of $\mathrm{M/E}$ corresponding by Galois theory to the subgroups $\mathrm{C}_{\tilde{\pi}}$ and $\mathrm{E}_{p;k}\mathrm{C}_{\tilde{\pi}}$ of $\mathrm{G(M/E)}$, respectively. It is easily seen that $\mathrm{M'/E}$ is a Galois extension with $\mathrm{G(M'/E)}$ satisfying the conditions of Lemma 3.2, and $\mathrm{E'/E}$ is a cyclic extension of degree π. This ensures that $c^\pi \in \mathrm{N(M'/E)}$. Also, it becomes clear that $\mathrm{M} = \mathrm{M}'\mathrm{M}_{ab}$, $\mathrm{M}' \cap \mathrm{M}_{ab} = \mathrm{E}'$ and $\mathrm{N}_{\mathrm{M}'}^{\mathrm{M}}(m') = m'^{\tilde{\pi}}$, for every $m' \in \mathrm{M}'$. These observations show that $c^{\pi^n} \in \mathrm{N(M/E)}$, so Lemma 3.3 is proved.

4. Proof of Theorems 3.1 and 1.1

Retaining notation as in Section 3, we first consider the special case in which R is an intermediate field of a finite Galois extension with a solvable Galois group. Our argument relies on Lemma 3.3 and the following lemma.

Lemma 4.1. *Under the hypotheses of Theorem 3.1, suppose that* M/E *is a Galois extension with a solvable Galois group* $\mathrm{G}(\mathrm{M}/\mathrm{E})$, *and* R *is an intermediate field of* M/E, *such that* $[\mathrm{R}\colon\mathrm{E}]$ *is a power of* p. *Then* $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\mathrm{R}_{\mathrm{ab}}/\mathrm{E})$.

Proof. Arguing by induction on $[\mathrm{M}\colon\mathrm{E}]$, one obtains from the conditions of Theorem 3.1 that it is sufficient to prove the lemma, assuming in addition that $\mathrm{N}(\mathrm{R}_1/\mathrm{E}_1) = \mathrm{N}(\mathrm{R}'/\mathrm{E}_1)$ whenever E_1 and R_1 are intermediate fields of M/E, such that $\mathrm{E}_1 \neq \mathrm{E}$, $\mathrm{E}_1 \subseteq \mathrm{R}_1$, $[\mathrm{R}_1\colon\mathrm{E}_1]$ is a power of p, and R' is the maximal abelian extension of E_1 in R_1. Suppose first that $\mathrm{R}_{\mathrm{ab}} \neq \mathrm{E}$. Then the inductive hypothesis, applied to the the pair $(\mathrm{E}_1, \mathrm{R}_1) = (\mathrm{R}_{\mathrm{ab}}, \mathrm{R})$, gives $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\mathrm{R}'/\mathrm{E})$, and since R' is a subfield of the maximal p-extension M_p of E in M, this enables one to obtain from the former part of (1.1) (ii) that $\mathrm{N}(\mathrm{R}'/\mathrm{E}) = \mathrm{N}(\mathrm{R}_{\mathrm{ab}}/\mathrm{E})$.

It remains to be seen that $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{E}^*$ in the special case of $\mathrm{R}_{\mathrm{ab}} = \mathrm{E}$. Our argument relies on the fact that $\mathrm{E}^*/\mathrm{N}(\mathrm{M}_{\mathrm{ab}}/\mathrm{E})$ is a group of exponent dividing $[\mathrm{M}_{\mathrm{ab}}\colon\mathrm{E}]$. Therefore, if $\mathrm{M}_p = \mathrm{E}$, then this exponent is not divisible by p. In view of the inclusion $\mathrm{N}(\mathrm{M}/\mathrm{E}) \subseteq \mathrm{N}(\mathrm{R}/\mathrm{E})$, $\mathrm{E}^*/\mathrm{N}(\mathrm{R}/\mathrm{E})$ is canonically isomorphic to a homomorphic image of $\mathrm{E}^*/\mathrm{N}(\mathrm{M}/\mathrm{E})$, so the condition $\mathrm{M}_p = \mathrm{E}$ ensures that the exponent $e(\mathrm{R}/\mathrm{E})$ of $\mathrm{E}^*/\mathrm{N}(\mathrm{R}/\mathrm{E})$ is also relatively prime to p. As $e(\mathrm{R}/\mathrm{E})$ divides $[\mathrm{R}\colon\mathrm{E}]$, this proves that $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{E}^*$.

Assume now that $\mathrm{R}_{\mathrm{ab}} = \mathrm{E}$ and $\mathrm{M}_p \neq \mathrm{E}$, denote by F_1 the maximal abelian extension of E in M_p, and by F_2 the intermediate field of M/E corresponding by Galois theory to some Sylow p-subgroup of $\mathrm{G}(\mathrm{M}/\mathrm{E})$. Put $\mathrm{R}_1 = \mathrm{RF}_1$, $\mathrm{R}_2 = \mathrm{RF}_2$ and $\mathrm{F}_3 = \mathrm{F}_1\mathrm{F}_2$. It follows from Galois theory and the equality $\mathrm{R}_{\mathrm{ab}} = \mathrm{E}$ that the compositum RM_p is a Galois extension of R with $\mathrm{G}((\mathrm{RM}_p)/\mathrm{R})$ canonically isomorphic to $\mathrm{G}(\mathrm{M}_p/\mathrm{E})$; in addition, it becomes clear that R_1 is the maximal abelian extension of R in RM_p. Thus it turns out that $[\mathrm{R}_1\colon\mathrm{R}] = [\mathrm{F}_1\colon\mathrm{E}]$, which means that $[\mathrm{R}_1\colon\mathrm{E}] = [\mathrm{R}\colon\mathrm{E}].[\mathrm{F}_1\colon\mathrm{E}]$. Observing that $[\mathrm{F}_2\colon\mathrm{E}]$ is not divisible by p, one also sees that $[\mathrm{R}_2\colon\mathrm{F}_2] = [\mathrm{R}\colon\mathrm{E}]$, $[(\mathrm{R}_1\mathrm{F}_2)\colon\mathrm{F}_2] = [\mathrm{R}_1\colon\mathrm{E}]$ and $[(\mathrm{RF}_3)\colon\mathrm{F}_2] = [\mathrm{R}_2\colon\mathrm{F}_2].[\mathrm{F}_3\colon\mathrm{F}_2]$. The concluding equality and the normality of F_3 over F_2 imply that $\mathrm{R}_2 \cap \mathrm{F}_3 = \mathrm{F}_2$. In view of Proposition 2.5 (iii) and Lemma 2.4, this leads to the conclusion that $\mathrm{N}(\mathrm{R}_2/\mathrm{E})\mathrm{N}(\mathrm{F}_3/\mathrm{E}) = \mathrm{N}(\mathrm{F}_2/\mathrm{E})$. Note also that $\mathrm{N}(\mathrm{F}_1/\mathrm{E}) = \mathrm{N}(\mathrm{R}_1/\mathrm{E})$. Indeed, it follows from Galois theory and the definition of M_p that M_p does not admit proper p-extensions in M, and by the inductive hypothesis, this yields $\mathrm{N}((\mathrm{RM}_p)/\mathrm{M}_p) = \mathrm{M}_p^*$. Hence, by the former part of (1.1) (ii) and the transitivity of norm mappings, we have $\mathrm{N}((\mathrm{RM}_p)/\mathrm{E}) = \mathrm{N}(\mathrm{M}_p/\mathrm{E}) = \mathrm{N}(\mathrm{F}_1/\mathrm{E})$. At the same time, since R_1 is the maximal abelian extension of R in RM_p, it turns out that $\mathrm{N}((\mathrm{RM}_p)/\mathrm{R}) = \mathrm{N}(\mathrm{R}_1/\mathrm{R})$, which implies that $\mathrm{N}((\mathrm{RM}_p)/\mathrm{E}) = \mathrm{N}(\mathrm{R}_1/\mathrm{E}) = \mathrm{N}(\mathrm{F}_1/\mathrm{E})$, as claimed. The obtained results and the inclusions $\mathrm{N}(\mathrm{R}_2/\mathrm{E}) \subseteq \mathrm{N}(\mathrm{R}/\mathrm{E})$ and $\mathrm{N}(\mathrm{F}_3/\mathrm{E}) \subseteq \mathrm{N}(\mathrm{F}_1/\mathrm{E})$, indicate that $\mathrm{N}(\mathrm{F}_2/\mathrm{E})$ is a subgroup of $\mathrm{N}(\mathrm{R}/\mathrm{E})\mathrm{N}(\mathrm{F}_1/\mathrm{E}) =$

$N(R/E)N(R_1/E) = N(R/E)$. As $E^*/N(R/E)$ and $E^*/N(F_2/E)$ are groups of finite relatively prime exponents, this means that that $N(R/E) = E^*$, so the proof of Lemma 4.1 is complete.

We are now in a position to prove Theorem 3.1 in the special case where R is an intermediate field of a finite Galois extension M/E with a solvable Galois group. It is clearly sufficient to establish our assertion under the additional hypothesis that $N_p(R_1/E_1)$ and $N(R_1/E_1)$ are related in accordance with Theorem 3.1 whenever E_1 and R_1 are extensions of E in R and M, respectively, such that $E_1 \neq E$ and $E_1 \subseteq R_1$. Suppose that $R \neq E$, put $\Phi = R_{ab,p}$, if $R_{\rm ab,p} \neq E$, and denote by Φ some proper extension of E in R of primary degree, otherwise (the existence of Φ in the latter case follows from Galois theory and the well-known fact that maximal subgroups of solvable finite groups are of primary indices). Also, let α be an element of $N_p(R/E) \cap N(R_{ab,p}/E)$, Φ' the maximal abelian p-extension of Φ in R, M' the compositum $\Phi M_{ab,p}$, k the maximal integer dividing $[M\colon E]$ and not divisible by p, and $\Phi^{*k} = \{z^k\colon\ z \in \Phi^*\}$. It is not difficult to see that $\Phi' \cap M' = \Phi$. Applying Proposition 2.5 (iii) or Lemma 2.4, depending on whether or not $\mathrm{Br}\,(\Phi)_p \neq \{0\}$, one obtains further that $\Phi^* = N(\Phi'/\Phi)N(M'/\Phi)$. Hence, by the inductive hypothesis and the inclusion $N_p(M/\Phi) \subseteq N_p(R/\Phi)$, Φ^{*k} is a subgroup of $N(R/\Phi).(N(M'/\Phi) \cap \Phi^{*k})$. Note also that Lemma 4.1 and the choice of Φ ensure the existence of an element $\xi \in \Phi$ of norm α over E. Taking now into account that $N(M'/E) \subseteq N(M_{\rm ab,p}/E)$, one obtains that $\alpha^k \in N(R/E)(N_p(M/E) \cap N(M_{ab,p}/E))$, and then deduces from Lemma 3.3 that $\alpha^k \in N(R/E)$. In view of the choice of α and k, this means that $\alpha \in N(R/E)$, which proves Theorem 3.1 in the discussed special case. In order to do the same in full generality, we need the following lemma.

Lemma 4.2. *Let* E *and* p *satisfy the conditions of Theorem 3.1, and let* R *be an intermediate field of a finite Galois extension* M/E, *such that* $G(M/E) = [G(M/E), G(M/E)]$. *Then* $N_p(R/E) \subseteq N(R/E)$.

Proof. It is clearly sufficient to consider only the special case of $R = M \neq E$ (and $\mathrm{Br}\,(E)_p \neq \{0\}$). Denote by E_p be the intermediate field of M/E corresponding by Galois theory to some Sylow p-subgroup of $G(M/E)$. Then p does not divide the degree $[E_p\colon E] := m_p$, so the condition $\mathrm{Br}\,(E)_p \neq \{0\}$ guarantees that $\mathrm{Br}\,(E_p)_p \neq \{0\}$. We first show that $E^* \subseteq N(M/E_p)$, assuming additionally that $\mathrm{char}\,(E) = p$ or E contains a primitive root of unity of degree $[M\colon E_p]$. As E is a quasilocal field, the nontriviality of $\mathrm{Br}\,(E_p)_p$ ensures that E_p admits local p-class field theory. Hence, by the former part of (1.1) (ii), it is sufficient to prove the inclusion $E^* \subseteq N(L/E_p)$, for an arbitrary cyclic extension L of E_p in M. By [20, Sect. 15.1, Proposition

b], this is equivalent to the assertion that the cyclic E_p-algebra $(\mathrm{L}/\mathrm{E}_p, \sigma, c)$ is isomorphic to the matrix E_p-algebra $M_n(\mathrm{E}_p)$, where $c \in \mathrm{E}^*$, $n = [\mathrm{L}\colon \mathrm{E}_p]$ and σ is an E_p-automorphism of L of order n. Since g.c.d. $([\mathrm{E}_p\colon \mathrm{E}], p) = 1$, the surjectivity of the natural homomorphism of $\mathrm{Br}\,(\mathrm{E})_p$ into $\mathrm{Br}\,(\mathrm{E}_p)_p$ implies that the corestriction homomorphism $\mathrm{cor}_{\,\mathrm{E}_p/\mathrm{E}}\colon \mathrm{Br}(\mathrm{E}_p) \to \mathrm{Br}\,(\mathrm{E})$ induces an isomorphism of $\mathrm{Br}\,(\mathrm{E}_p)_p$ on $\mathrm{Br}\,(\mathrm{E})_p$ (cf. [27, Theorem 2.5]). Observe now that $\mathrm{cor}_{\,\mathrm{E}_p/\mathrm{E}}$ maps the similarity class $[(\mathrm{L}/\mathrm{E}_p, \sigma, c)]$ into $[(\widetilde{\mathrm{L}}/\mathrm{E}, \tilde{\sigma}, c)]$, for some cyclic p-extension $\widetilde{\mathrm{L}}$ of E in M (and a suitably chosen generator $\tilde{\sigma}$ of $\mathrm{G}(\widetilde{\mathrm{L}}/\mathrm{E})$). Since E contains a primitive root of unity of degree $[\mathrm{M}\colon \mathrm{E}_p]$ or $\mathrm{char}\,(\mathrm{E}) = p$, this can be obtained by applying the projection formula (cf. [16, Proposition 3 (i)] and [27, Theorem 3.2]), as well as Kummer's theory and its analogue, due to Witt, for finite abelian p-extensions over a field of characteristic p, (see, for example, [13, Ch. 7, Sect. 3]). As $\mathrm{G}(\mathrm{M}/\mathrm{E}) = [\mathrm{G}(\mathrm{M}/\mathrm{E}), \mathrm{G}(\mathrm{M}/\mathrm{E})]$, or equivalently, $\mathrm{M}_{\mathrm{ab}} = \mathrm{E}$, the obtained result shows that $\widetilde{\mathrm{L}} = \mathrm{E}_p$ and $[(\widetilde{\mathrm{L}}/\mathrm{E}, \tilde{\sigma}, c)] = 0$ in $\mathrm{Br}\,(\mathrm{E})$. Furthermore, it becomes clear that $[(\mathrm{L}/\mathrm{E}_p, \sigma, c)] = 0$ in $\mathrm{Br}\,(\mathrm{E}_p)$, i.e. $c \in \mathrm{N}(\mathrm{L}/\mathrm{E}_p)$, which proves the inclusion $\mathrm{E}^* \subseteq \mathrm{N}(\mathrm{M}/\mathrm{E}_p)$. Since $\mathrm{N}_{\mathrm{E}}^{\mathrm{E}_p}(c) = c^{m_p}$, one also sees that $c^{m_p} \in \mathrm{N}(\mathrm{M}/\mathrm{E})$, for each $c \in \mathrm{E}^*$.

Suppose now that $p \neq \mathrm{char}\,(\mathrm{E})$, fix a primitive root of unity $\varepsilon \in \mathrm{M}_{\mathrm{sep}}$ of degree $[\mathrm{M}\colon \mathrm{E}_p]$, and put $\Phi(\varepsilon) = \Phi'$, for every intermediate field Φ of M/E, and $\mathrm{H}^{m_p} = \{h^{m_p}\colon h \in \mathrm{H}\}$, for each subgroup H of M'^*. As E'/E is an abelian extension, our assumption on $\mathrm{G}(\mathrm{M}/\mathrm{E})$ ensures that $\mathrm{E}' \cap \mathrm{M} = \mathrm{E}$, and by Galois theory, this means that M'/E' is a Galois extension with $\mathrm{G}(\mathrm{M}'/\mathrm{E}')$ canonically isomorphic to $\mathrm{G}(\mathrm{M}/\mathrm{E})$. Thus it becomes clear from the previous considerations that $\mathrm{E}'^{*m_p} \subseteq \mathrm{N}(\mathrm{M}'/\mathrm{E}')$ and $\mathrm{N}(\mathrm{E}'/\mathrm{E})^{m_p} \subseteq \mathrm{N}(\mathrm{M}'/\mathrm{E}) \subseteq \mathrm{N}(\mathrm{M}/\mathrm{E})$. Our argument also shows that $\mathrm{M} \cap \mathrm{E}'_p = \mathrm{E}_p$, and since E_p is p-quasilocal, it enables one to deduce from Proposition 2.5 (iii), the former part of (1.1) (ii), and Lemma 2.2 that $\mathrm{N}(\mathrm{M}/\mathrm{E}_p)\mathrm{N}(\mathrm{E}'_p/\mathrm{E}_p) = \mathrm{E}_p^*$. Hence, by the transitivity of norm mappings, $\mathrm{N}(\mathrm{M}/\mathrm{E})\mathrm{N}(\mathrm{E}'_p/\mathrm{E}) = \mathrm{N}(\mathrm{E}_p/\mathrm{E})$. These observations prove the inclusions $\mathrm{E}^{*m_p^2} \subseteq \mathrm{N}(\mathrm{E}_p/\mathrm{E})^{m_p} \subseteq \mathrm{N}(\mathrm{M}/\mathrm{E})^{m_p}.\mathrm{N}(\mathrm{E}'/\mathrm{E})^{m_p} \subseteq \mathrm{N}(\mathrm{M}/\mathrm{E})$. This, combined with the fact that p does not divide m_p and the exponent of $\mathrm{E}^*/\mathrm{N}(\mathrm{M}/\mathrm{E})$ divides $[\mathrm{M}\colon \mathrm{E}]$, indicates that $\mathrm{E}^{*m_p} \subseteq \mathrm{N}(\mathrm{M}/\mathrm{E})$ and so completes the proof of Lemma 4.2.

It is now easy to accomplish the proof of Theorem 3.1. Assume that M_0 is the maximal Galois extension of E in M with a solvable Galois group, and also, that μ_p, m_p and ρ_p are the maximal integers not divisible by p and dividing $[\mathrm{M}_0\colon \mathrm{E}]$, $[\mathrm{M}\colon \mathrm{E}]$ and $[\mathrm{R}\colon \mathrm{E}]$, respectively. Applying Lemma 3.3 to M_0/E and Lemma 4.2 to M/M_0, one obtains that $\mathrm{E}^{*\mu_p} \subseteq \mathrm{N}(\mathrm{M}_0/\mathrm{E})$ and $\mathrm{M}_0^{*\bar{m}_p} \subseteq \mathrm{N}(\mathrm{M}/\mathrm{M}_0)$, where $\bar{m}_p = m_p/\mu_p$. Hence, by the norm identity $\mathrm{N}_{\mathrm{E}}^{\mathrm{M}} = \mathrm{N}_{\mathrm{E}}^{\mathrm{M}_0} \circ \mathrm{N}_{\mathrm{M}_0}^{\mathrm{M}}$, we have $\mathrm{E}^{*m_p} \subseteq \mathrm{N}(\mathrm{M}_0/\mathrm{E})^{\bar{m}_p} \subseteq \mathrm{N}(\mathrm{M}/\mathrm{E})$. Since $\mathrm{E}^{*[\mathrm{R}:\mathrm{E}]} \subseteq \mathrm{N}(\mathrm{R}/\mathrm{E})$, $\mathrm{N}(\mathrm{M}/\mathrm{E}) \subseteq \mathrm{N}(\mathrm{R}/\mathrm{E})$ and

g.c.d. $(m_p, [\mathrm{R}\colon \mathrm{E}]) = \rho_p$, this means that $\mathrm{E}^{*\rho_p} \subseteq \mathrm{N}(\mathrm{R}/\mathrm{E})$, so Theorem 3.1 is proved.

Remark 4.3. Lemma 4.2 remains valid (with a slightly modified proof), if the condition on $\mathrm{G}(\mathrm{M}/\mathrm{E})$ is replaced by the one that p does not divide the index $|\mathrm{G}(\mathrm{M}/\mathrm{E})\colon [\mathrm{G}(\mathrm{M}/\mathrm{E}), \mathrm{G}(\mathrm{M}/\mathrm{E})]|$. Note also that Lemma 3.3 can be deduced from (3.3) and this generalization of Lemma 4.2, which allows us to skip Lemma 3.2 and shorten the proof of Theorem 3.1. When G_{E} is a prosolvable group, however, the inclusion of Lemma 3.2 enables us to deduce the theorem from Proposition 2.5 (iii), fundamentals of Galois theory, basic properties of cyclic algebras and well-known elementary facts concerning solvable finite groups. The prosolvability of G_{E} is guaranteed, if E possesses a Henselian discrete valuation (cf. [3, Corollary 2.5 and Proposition 3.1]).

Proof of Theorem 1.1. Since Theorem 1.1 (i) is a special case of Theorem 3.1, it is sufficient to prove Theorem 1.1 (ii). Let E be a quasilocal field and R, $\Phi(\mathrm{R})$ be finite extensions of E in $\mathrm{E}_{\mathrm{sep}}$, such that $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\Phi(\mathrm{R})/\mathrm{E})$ and $\Phi(\mathrm{R})/\mathrm{E}$ is abelian. Applying Lemmas 2.2 and 2.3, one reduces the proof of Theorem 1.1 (ii) to the special case in which $\mathrm{Br}\,(\mathrm{E})_p \neq \{0\}$, when p ranges over the set Π of prime numbers dividing $[\Phi(\mathrm{R})\colon \mathrm{E}]$. Let Λ be the normal closure of R in $\mathrm{E}_{\mathrm{sep}}$ over E, and for each $p \in \Pi$, let $\Phi(\mathrm{R})_p$ be the maximal p-extension of E in $\Phi(\mathrm{R})$, H_p be a Sylow p-subgroup of $\mathrm{G}(\Lambda/\mathrm{R})$, G_p a Sylow p-subgroup of $\mathrm{G}(\Lambda/\mathrm{E})$ including H_p, R_1 and E_1 the intermediate fields of Λ/E corresponding by Galois theory to H_p and G_p, respectively. Note first that $\mathrm{R}_{\mathrm{ab,p}}$ is a subfield of $\Phi(\mathrm{R})_p$. Indeed, the nontriviality of $\mathrm{Br}\,(\mathrm{E})_p$ and the PQL-property of E ensure the availability of a local p-class field theory on E, so our assertion follows from the fact that $\mathrm{N}(\Phi(\mathrm{R})/\mathrm{E}) = \mathrm{N}(\mathrm{R}/\mathrm{E}) \subseteq \mathrm{N}(\mathrm{R}_{\mathrm{ab,p}}/\mathrm{E})$ (whence, by Lemma 2.2, we have $\mathrm{N}(\Phi(\mathrm{R})_p/\mathrm{E}) \subseteq \mathrm{N}(\mathrm{R}_{\mathrm{ab,p}}/\mathrm{E})$). It is easily verified that p does not divide $[\mathrm{R}_1\colon \mathrm{R}][\mathrm{E}_1\colon \mathrm{E}]$ and $\mathrm{R}_{\mathrm{ab,p}}\mathrm{E}_1 = (\Phi(\mathrm{R})_p\mathrm{E}_1) \cap \mathrm{R}_1$. One also sees that $\mathrm{Br}\,(\mathrm{E}_1)_p \neq \{0\}$ (cf. [20, Sect. 13.4]). As E is quasilocal, this indicates that E_1 admits local p-class field theory, so it follows from the former part of (1.1) (ii) that $\mathrm{N}((\mathrm{R}_{\mathrm{ab,p}}\mathrm{E}_1)/\mathrm{E}_1) = \mathrm{N}((\Phi(\mathrm{R})_p\mathrm{E}_1)/\mathrm{E}_1)\mathrm{N}(\mathrm{R}_1/\mathrm{E}_1)$. Our argument also proves that $\mathrm{N}((\mathrm{R}_{\mathrm{ab,p}}\mathrm{E}_1)/\mathrm{E}) = \mathrm{N}((\Phi(\mathrm{R})_p\mathrm{E}_1)/\mathrm{E})\mathrm{N}(\mathrm{R}_1/\mathrm{E}) \subseteq (\mathrm{N}(\Phi(\mathrm{R})_p/\mathrm{E})\mathrm{N}(\mathrm{R}/\mathrm{E}) \cap \mathrm{N}(\mathrm{E}_1/\mathrm{E})) =$ $\mathrm{N}(\Phi(\mathrm{R})_p\mathrm{E}_1)/\mathrm{E})$. On the other hand, the inclusion $\mathrm{R}_{\mathrm{ab,p}} \subseteq \Phi(\mathrm{R})_p$ implies that $\mathrm{N}((\Phi(\mathrm{R})_p\mathrm{E}_1)/\mathrm{E}) \subseteq \mathrm{N}((\mathrm{R}_{\mathrm{ab,p}}\mathrm{E}_1)/\mathrm{E})$, so it turns out that $\mathrm{N}(\Phi(\mathrm{R})_p\mathrm{E}_1)/\mathrm{E}) =$ $\mathrm{N}((\mathrm{R}_{\mathrm{ab,p}}\mathrm{E}_1)/\mathrm{E})$ and the quotient group $\mathrm{N}(\mathrm{R}_{\mathrm{ab,p}}/\mathrm{E})/\mathrm{N}(\Phi(\mathrm{R})_p/\mathrm{E})$ is of exponent e_p dividing $[\mathrm{E}_1\colon \mathrm{E}]$. Since e_p divides $[\Phi(\mathrm{R})\colon \mathrm{E}]$ and p does not divide $[\mathrm{E}_1\colon \mathrm{E}]$, this means that $e_p = 1$, i.e. $\mathrm{N}(\Phi(\mathrm{R})_p/\mathrm{E}) = \mathrm{N}(\mathrm{R}_{\mathrm{ab,p}}/\mathrm{E})$ (and $\Phi(\mathrm{R})_p = \mathrm{R}_{\mathrm{ab,p}}$), for each $p \in \Pi$. Let now p' be an arbitrary prime number. It is clear from the inclusion $\mathrm{R}_{ab,p'} \subseteq \mathrm{R}$ that $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\Phi(\mathrm{R})/\mathrm{E}) \subseteq \mathrm{N}(\mathrm{R}_{ab,p'}/\mathrm{E})$ and $\mathrm{E}^*/\mathrm{N}(\mathrm{R}_{ab,p'}/\mathrm{E})$ is a homomorphic image of $\mathrm{E}^*/\mathrm{N}(\Phi(\mathrm{R})/\mathrm{E})$, so it follows from Lemma 2.2 that

$\mathrm{E}^*/\mathrm{N}(\mathrm{R}_{ab,p'}/\mathrm{E})$ is a group of exponent dividing $[\Phi(\mathrm{R})\colon \mathrm{E}]$. It is now easy to see that $\mathrm{N}(\mathrm{R}_{ab,p'}/\mathrm{E}) = \mathrm{E}^*$ whenever $p' \notin \Pi$, and to conclude that $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\mathrm{R}_{ab}/\mathrm{E})$, as claimed by Theorem 1.1 (ii).

Remark 4.4. (i) The conditions of Theorem (i) are in force, if E is a field with local class field theory in the sense of Neukirch-Perlis [19], i.e. if the triple $(\mathrm{G}_{\mathrm{E}}, \{\mathrm{G}(\mathrm{E}_{\mathrm{sep}}/\mathrm{F}), \mathrm{F} \in \Sigma\}, \mathrm{E}^*_{\mathrm{sep}})$ is an Artin-Tate class formation (cf. [2, Ch. XIV]), where Σ is the set of finite extensions of E in $\mathrm{E}_{\mathrm{sep}}$. Then the assertion of Theorem 1.1 (i) is contained in [2, Ch. XIV, Theorem 7]; in particular, it applies to any p-adically closed field and includes (1.1) (i) as a special case (see [21, Theorem 3.1 and Lemma 2.9] and [26, Ch. XIII, Proposition 6], respectively).
(ii) Let us note that the class of fields satisfying the conditions of Theorem 1.1 (i) is larger than the one studied in [19]. More precisely, for every divisible abelian torsion group T, there exists a quasilocal field $\mathrm{E}(\mathrm{T})$ of this type, such that $\mathrm{Br}\,(\mathrm{E}(\mathrm{T}))$ is isomorphic to T and all finite groups are realizable as Galois groups over $\mathrm{E}(\mathrm{T})$ (this will be proved elsewhere), whereas the Brauer groups of the fields considered in [19] embed in $\mathbb{Q}/\mathrm{Z}$. These properties of $\mathrm{E}(\mathrm{T})$ indicate that it is strictly quasilocal if and only if the p-components of T are nontrivial, for all prime numbers p.
(iii) It follows at once from (1.1) (iii) and Theorem 1.1 (ii) that $\mathrm{N}(\mathrm{R}/\mathrm{E}) = \mathrm{N}(\mathrm{R}_{\mathrm{ab}}/\mathrm{E})$ whenever E is quasilocal and algebraic over a global field E_0. In this case, G_{E} is prosolvable and E satisfies the conditions of Theorem 1.1 (i) as well (see [9, Proposition 2.7] and the references there).

5. Proof of Theorem 1.2

In this Section we characterize (and prove the existence of) Henselian discrete valued strictly quasilocal fields with the properties required by Theorem 1.2. In what follows, $\overline{\mathrm{P}}$ is the set of prime numbers, and for each field E, $\mathrm{P}_0(\mathrm{E})$ is the subset of those $p \in \overline{\mathrm{P}}$, for which E contains a primitive p-th root of unity, or else, $p = \mathrm{char}\,(\mathrm{E})$. Also, we denote by $\mathrm{P}_1(\mathrm{E})$ the subset of those $p' \in (\overline{\mathrm{P}} \setminus \mathrm{P}_0(\mathrm{E}))$, for which $\mathrm{E}^* \neq \mathrm{E}^{*p'}$, and put $\mathrm{P}_2(\mathrm{E}) = \overline{\mathrm{P}} \setminus (\mathrm{P}_0(\mathrm{E}) \cup \mathrm{P}_1(\mathrm{E}))$. Every finite extension L of a field K with a Henselian valuation v is considered with its valuation extending v, this prolongation is also denoted by v (unless stated otherwise), and $e(\mathrm{L}/\mathrm{K})$ denotes the ramification index of L/K. Our starting point is the following statement (proved in [6]):

(5.1) With assumptions being as above, if v is discrete, then the following conditions are equivalent:
(i) K is strictly quasilocal;
(ii) The residue field $\widehat{\mathrm{K}}$ of (K, v) is perfect, the absolute Galois group $\mathrm{G}_{\widehat{\mathrm{K}}}$ is metabelian of cohomological p-dimension $\mathrm{cd}_p(\mathrm{G}_{\mathrm{K}}) = 1$, for each $p \in \overline{\mathrm{P}}$, and

$P_0(\widetilde{L}) \subseteq P(\widetilde{L})$, for every finite extension $\widetilde{L}$ of $\widehat{K}$.

When these conditions are in force, K is a nonreal field (cf. [14, Theorem 3.16]), $P_0(K) \setminus \{\mathrm{char}(\widehat{K})\} = P_0(\widehat{K}) \setminus \{\mathrm{char}(\widehat{K})\}$, and the following is true:

(5.2) (i) $\mathrm{Br}\,(\widetilde{L}) = \{0\}$ and $\mathrm{Br}\,(L)_p$ is isomorphic to the quasicyclic p-group $\mathbb{Z}(p^\infty)$, for every finite extension L/K and each $p \in P(\widehat{K})$ (apply [25, Ch. II, Proposition 6 (b)] and Scharlau's generalization of Witt's theorem [23]); in particular, the natural homomorphism $\mathrm{Br}\,(K)_p \to \mathrm{Br}\,(L)_p$ is surjective;
(ii) If R is a finite extension of K in K_{sep}, such that $[R\colon K]$ is not divisible by $\mathrm{char}\,(\widehat{K})$ or any $p \in P_2(\widehat{K})$, then R is presentable as a compositum of subextensions of K of primary degrees; furthermore, if $[R\colon K]$ is not divisible by $\mathrm{char}\,(\widehat{K})$ or any $p \in (P_1(\widehat{K}) \cup P_2(\widehat{K}))$, then the normal closure of R in K_{sep} over K has a nilpotent Galois group (apply [10, (3.3)] and Galois theory);
(iii) The group G_K is pronilpotent in case $\mathrm{char}\,(\widehat{K}) = 0$ and $P_0(\widehat{K}) = \overline{P}$.

The following result (proved in [10]) sheds light on the norm groups of finite separable extensions of a field K subject to the restrictions of (5.1). It shows that the conclusion of (1.1) (i) is generally valid if and only if $P(\widehat{K}) = \overline{P}$, i.e. $\widehat{K}$ is quasifinite.

Proposition 5.1. *Assume that* (K, v) *is a Henselian discrete valued strictly quasilocal field, and* R *is a finite extension of* K *in* K_{sep}. *Then* R/K *possesses an intermediate field* R_1 *such that:*
(i) *The sets of prime divisors of* $e(R_1/K)$, $[\widehat{R}_1\colon\widehat{K}]$, $[\widehat{R}\colon\widehat{R}_1]$ *and* $[R\colon R_1]$ *are included in* $P_1(\widehat{K})$, $\overline{P} \setminus P(\widehat{K})$, $P(\widehat{K})$ *and* $P_0(\widehat{K}) \cup P_2(\widehat{K})$, *respectively;*
(ii) $N(R/K) = N((R_{ab}R_1)/K)$ *and* $K^*/N(R/K)$ *is isomorphic to the direct sum* $G(R_{ab}/K) \times (K^*/N(R_1/K))$*;*
(iii) $K^*/N(R/K)$ *is of order* $[R_{ab}\colon K][R_1\colon K] = [(R_{ab}R_1)\colon K]$.

Our next result characterizes the fields singled out by Theorem 1.2 (i)-(ii) in the class of strictly quasilocal fields with Henselian discrete valuations:

Proposition 5.2. *For a strictly quasilocal field* K *with a Henselian discrete valuation* v, *the following conditions are equivalent:*
(i) G_K *and the finite extensions of* K *have the properties required by Theorem 1.2;*
(ii) $\mathrm{char}\,(\widehat{K}) = 0$, $P_0(\widehat{K}) = P(\widehat{K}) \neq \overline{P}$ *and* $P_1(\widehat{K}) = \overline{P} \setminus P_0(\widehat{K})$.
When this occurs, every finite extension R *of* K *in* K_{sep} *is presentable as a compositum* $R = R_0R_1$, *where* R_1 *is determined in accordance with Proposition 5.1 (i) and (ii), and* R_0 *is an intermediate field of* R/K *of degree* $[R_0\colon K] = [R\colon R_1]$. *Moreover, the Galois group of the normal closure* $\widetilde{R}$ *of* R *in* K_{sep} *over* K *is nilpotent if and only if* $R = R_0$.

Proof. The implication (ii) $\to$ (i) follows from Proposition 5.1 and the fact that R_1 is defectless over K [28, Propositions 2.2 and 3.1]. The concluding assertions of Proposition 5.2 are implied by (5.2) (ii), so we assume further that condition (i) is in force. Let π be a generator of the maximal ideal of the valuation ring of (K, v). It is easily deduced from Proposition 5.1 that if $p \in \mathrm{P}_2(\widehat{\mathrm{K}})$ or $p = \mathrm{char}\,(\widehat{\mathrm{K}})$ and $p \notin \mathrm{P}_0(\mathrm{K})$, then the root field, say M_π, of the binomial $\mathrm{X}^p - \pi$ satisfies the equality $\mathrm{N}(\mathrm{M}_\pi/\mathrm{K}) = \mathrm{N}(\mathrm{M}_{\pi,\mathrm{ab}}/\mathrm{K})$. At the same time, it follows from (2.2) that $\mathcal{G}(\mathrm{M}_\pi/\mathrm{K})$ is nonabelian and isomorphic to a semidirect product of a group of order p by a cyclic group of order dividing $p-1$. This indicates that $\mathcal{G}(\mathrm{M}_\pi/\mathrm{K})$ is nonnilpotent. The obtained results contradict condition (i), and thereby, prove that $\mathrm{P}_0(\widehat{\mathrm{K}}) \cup \mathrm{P}_1(\widehat{\mathrm{K}}) = \overline{\mathrm{P}}$. Hence, by Galois theory, (1.1) (i) and condition (i), $\widehat{\mathrm{K}}$ is an infinite field. It remains to be seen that $\mathrm{char}\,(\widehat{\mathrm{K}}) = 0$ and $\mathrm{P}_0(\widehat{\mathrm{K}}) \neq \overline{\mathrm{P}}$. Suppose that $\mathrm{char}\,(\widehat{\mathrm{K}}) = q > 0$ and $q \in \mathrm{P}_0(\mathrm{K})$. Then condition (i), statement (5.1) and the infinity of $\widehat{\mathrm{K}}$ imply the existence of a primitive p-th root of unity in $\widehat{\mathrm{K}}$, for at least one prime number $p \neq q$. In addition, it becomes clear that there exists a cyclic inertial extension L_p of K in $\mathrm{K}_{\mathrm{sep}}$ of degree p. Let v_p be the valuation of L_p extending v. It is easily obtained from Galois theory (cf. [15, Ch. VIII, Theorem 20]) and the Henselian property of v that L_p has a normal basis B_p over K, such that $v_p(b) = 0$, for all $b \in \mathrm{B}_p$. Denote by B'_p the polynomial set $\{\mathrm{X}^q - \mathrm{X} - b\pi^{-1} \colon b \in \mathrm{B}_p\}$, if $\mathrm{char}\,(\mathrm{K}) = q$, and put $\mathrm{B}'_p = \{\mathrm{X}^q - (1 + b\pi) \colon b \in \mathrm{B}_p\}$, in the mixed-characteristic case. It follows from the Artin-Schreier theorem, Capelli's criterion (cf. [15, Ch. VIII, Sect. 9]) and the Henselian property of v_p that B'_p consists of irreducible polynomials over L_p. Furthermore, one obtains from Kummer's theory (and the assumption that $q \in \mathrm{P}_0(\mathrm{K})$) that the root field L'_p of B'_p over L_p is a Galois extension of K of degree $q^p p$. It follows from the definition of L'_p that the Sylow q-subgroup $\mathcal{G}(\mathrm{L}'_p/\mathrm{L}_p)$ of $\mathcal{G}(\mathrm{L}'_p/\mathrm{K})$, is normal and elementary abelian. At the same time, it is clear from the choice of B_p that $\mathcal{G}(\mathrm{L}'_p/\mathrm{L}_p)$ possesses maximal subgroups that are not normal in $\mathcal{G}(\mathrm{L}'_p/\mathrm{E})$. These properties of $\mathcal{G}(\mathrm{L}'_p/\mathrm{L}_p)$ indicate that $\mathcal{G}(\mathrm{L}'_p/\mathrm{E})$ is nonnilpotent. On the other hand, since q and p lie in $\mathrm{P}(\widehat{\mathrm{K}})$, Theorem 3.1 and the latter assertion of (5.2) (i) show that $\mathrm{N}(\mathrm{L}'_p/\mathrm{K}) = \mathrm{N}(\mathrm{L}'_{p,\mathrm{ab}}/\mathrm{K})$. Thus the hypothesis that $\mathrm{char}\,(\widehat{\mathrm{K}}) \neq 0$ leads to a contradiction with condition (i), so the proof of Proposition 5.2 can be accomplished by applying (5.2) (iii).

Corollary 5.3. *Let* (K, v) *be a Henselian discrete valued field satisfying the conditions of Proposition 5.2,* ε_p *a primitive* p*-th root of unity in* $\mathrm{K}_{\mathrm{sep}}$*, for each* $p \in \overline{\mathrm{P}}$*, and* $[\mathrm{K}(\varepsilon_p) \colon \mathrm{K}] = \gamma_p$ *in case* $p \in (\overline{\mathrm{P}} \setminus \mathrm{P}(\widehat{\mathrm{K}}))$*. Then each finite extension* L *of* K *in* $\mathrm{K}_{\mathrm{sep}}$ *is subject to the following alternative:*

(i) $\mathcal{G}_{\mathrm{L}}$ *and finite extensions of* L *have the properties required by Theorem 1.2;* (ii)

$\mathrm{G_L}$ *is pronilpotent.*

The latter occurs if and only if the set $\Gamma(\mathrm{K}) = \{\gamma_p \colon\ p \in (\overline{\mathrm{P}} \setminus \mathrm{P}(\widehat{\mathrm{K}}))\}$ *is bounded and* L *contains as a subfield the inertial extension of* K *in* $\overline{\mathrm{K}}$ *of degree equal to the least common multiple of the elements of* $\Gamma(\mathrm{K})$ *.*

Proof. Statement (5.1) and our assumptions guarantee that $\mathrm{P}_0(\widehat{\mathrm{L}}) \cup \mathrm{P}_1(\widehat{\mathrm{L}}) = \overline{\mathrm{P}}$, so the stated alternative is contained in (5.2) (iii).

Our next result supplements Proposition 5.1 and combined with Proposition 5.2, proves Theorem 1.2.

Proposition 5.4. *Let* P_0 *,* P_1 *,* P_2 *and* P *be subsets of the set* $\overline{\mathrm{P}}$ *of prime numbers, such that* $\mathrm{P}_0 \cup \mathrm{P}_1 \cup \mathrm{P}_2 = \overline{\mathrm{P}}$ *,* $2 \in \mathrm{P}_0$ *,* $\mathrm{P}_i \cap \mathrm{P}_j = \phi \colon\ 0 \le i < j \le 2$ *, and* $\mathrm{P}_0 \subseteq \mathrm{P} \subseteq (\mathrm{P}_0 \cup \mathrm{P}_2)$ *. For each* $p \in (\mathrm{P}_1 \cup \mathrm{P}_2)$ *, let* γ_p *be an integer* ≥ 2 *dividing* $p-1$ *and not divisible by any element of* $\overline{\mathrm{P}} \setminus \mathrm{P}$ *. Assume also that* $\gamma_p \ge 3$ *in case* $p \in (\mathrm{P}_2 \setminus \mathrm{P})$ *. Then there exists a Henselian discrete valued field* (K, v) *satisfying the following conditions:*

(i) K *is strictly quasilocal with* $\mathrm{P}(\widehat{\mathrm{K}}) = \mathrm{P}$ *and* $\mathrm{P}_j(\widehat{\mathrm{K}}) = \mathrm{P}_j \colon\ j = 0, 1, 2$ *;*

(ii) *For each* $p \in (\mathrm{P}_1 \cup \mathrm{P}_2)$ *,* γ_p *equals the degree* $[\mathrm{K}(\varepsilon_p) \colon \mathrm{K}]$ *, where* ε_p *is a primitive* p *-th root of unity in* $\mathrm{K_{sep}}$ *.*

Proof. Denote by G_1 and G_0 the topological group products $\prod_{p \in \mathrm{P}} \mathbb{Z}_p$ and $\prod_{p \in (\overline{\mathrm{P}} \setminus \mathrm{P})} \mathbb{Z}_p$ (i.e. $\mathrm{G}_0 = \{1\}$ in case $\mathrm{P} = \overline{\mathrm{P}}$), respectively, and fix an algebraic closure $\overline{\mathbb{Q}}$ of the field of rational numbers as well as a primitive p-th root of unity $\varepsilon_p \in \overline{\mathbb{Q}}$, for each $p \in \overline{\mathrm{P}}$. Also, let E_0 be a subfield of $\overline{\mathbb{Q}}$, such that $\mathrm{P}_0(\mathrm{E}_0) = \mathrm{P}_0$, $\mathrm{P}(\mathrm{E}_0) = \mathrm{P}$, $[\mathrm{E}_0(\varepsilon_p) \colon \mathrm{E}_0] = \gamma_p \colon\ p \in (\overline{\mathrm{P}} \setminus \mathrm{P}_0)$, and $\mathrm{G}_{\mathrm{E}_0} \cong \mathrm{G}_1$ (the existence of E_0 is guaranteed by [6, Lemma 3.5]). Suppose further that φ is a topological generator of $\mathrm{G}_{\mathrm{E}_0}$, and for each $p \in (\overline{\mathrm{P}} \setminus \mathrm{P})$, δ_p is a primitive γ_p-th root of unity in $\mathbb{Z}_p$, s_p and t_p are integers, such that $\varphi(\varepsilon_p) = \varepsilon_p^{s_p}$, $t_p - \delta_p \in p\mathbb{Z}_p$, and $0 \le s_p, t_p \le (p-1)$. Assume also that the roots δ_p are taken so that $t_p = s_p$ if and only if $p \in \mathrm{P}_1$. Regarding $\mathbb{Z}_p$ as as subgroup of G_0 , whenever $p \in (\overline{\mathrm{P}} \setminus \mathrm{P})$, consider the topological semidirect product $\mathrm{G} = \mathrm{G}_0 \times \mathrm{G}_{\mathrm{E}_0}$, defined by the rule $\varphi \lambda_p \varphi^{-1} = \delta_p \lambda_p \colon\ p \in (\overline{\mathrm{P}} \setminus \mathrm{P})$, $\lambda_p \in \mathbb{Z}_p$. It has been proved in [6, Sect. 3] that there exists a Henselian discrete valued strictly quasilocal field (K, v) , such that $\mathrm{G}_{\widehat{\mathrm{K}}}$ is continuously isomorphic to G , E_0 is a subfield of $\widehat{\mathrm{K}}$, and E_0 is algebraically closed in $\widehat{\mathrm{K}}$. In particular, this implies that $\mathrm{P}_0(\widehat{\mathrm{K}}) = \mathrm{P}_0$, $\mathrm{P}(\widehat{\mathrm{K}}) = \mathrm{P}$ and $[\mathrm{K}(\varepsilon_p) \colon \mathrm{K}] = \gamma_p \colon\ p \in (\overline{\mathrm{P}} \setminus \mathrm{P}_0)$. Applying finally (2.2) (ii), one concludes that $\mathrm{P}_1(\widehat{\mathrm{K}}) = \mathrm{P}_1$ and so completes the proof of Proposition 5.4.

Corollary 5.5. *There exists a set* $\{(\mathrm{K}_n, v_n) \colon\ n \in \mathbb{N} \cup \infty\}$ *of Henselian discrete valued strictly quasilocal fields satisfying the following conditions:*

(i) *The absolute Galois group of a finite extension* R_n *of* K_n *is pronilpotent if and only if* $n \in \mathbb{N}$ *and* R_n *contains as a subfield an inertial extension of* K_n *of degree* n *;*
(ii) *Finite extensions of* K_n *are subject to the alternative described in Theorem 1.2, provided that* $n \geq 2$ *.*

Proof. This follows at once from Corollary 5.3 and Proposition 5.4.

Corollary 5.6. *Let* P_0 *and* P *be subsets of the set* $\overline{\mathrm{P}}$ *of prime numbers, such that* $2 \in \mathrm{P}_0$ *and* $\mathrm{P}_0 \subseteq \mathrm{P}$ *. Then there exists a strictly quasilocal nonreal field* E *such that:*
(i) $\mathrm{P}_0(\mathrm{E}) = \mathrm{P}_0$ *and* $\{p \in \overline{\mathrm{P}}\colon \mathrm{cd}_p(\mathrm{G}_\mathrm{E}) \neq 0\} = \mathrm{P}$ *;*
(ii) *If* $\mathrm{P} \neq \mathrm{P}_0$ *, then* G_E *is nonnilpotent and finite extensions of* E *are subject to the alternative described by Theorem 1.2.*

Proof. Proposition 5.4 implies the existence of a Henselian discrete valued strictly quasilocal field (K, v), such that char $(\widehat{\mathrm{K}}) = 0$, $\mathrm{P}_0(\widehat{\mathrm{K}}) = \mathrm{P}_0$, $\mathrm{P}_1(\widehat{\mathrm{K}}) = \overline{\mathrm{P}} \setminus \mathrm{P}_0$, and for each $p \in \mathrm{P}_1(\widehat{\mathrm{K}})$, the extension of K in $\mathrm{K}_{\mathrm{sep}}$ obtained by adjoining a primitive p-th root of unity is of even degree. By [3, Proposition 3.1], G_K is a prosolvable group, which means that it possesses a closed Hall pro- P-subgroup H_P. Note finally that one can take as E the intermediate field of $\mathrm{K}_{\mathrm{sep}}/\mathrm{K}$ corresponding by Galois theory to H_P.

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